Hybridized discrete model for the anisotropic Kardar-Parisi-Zhang equation

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We discuss a hybridized discrete model for the anisotropic Kardar-Parisi-Zhang (KPZ) equation. We apply a restricted solid-on-solid rule in one direction and a ballistic growth rule in the other direction such that the model has anisotropic KPZ nonlinearities with opposite signs. The surface width of the model shows that it belongs to the Edwards-Wilkinson class. The directional height-height correlation functions confirm that the values of the dynamic exponent are independent of the directions even for the anisotropic case. The negative exponents in higher dimensions are also discussed. [S1063-651X(98)14806-7]

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Recently, the surface fluctuation of driven growth models has been intensively studied by using both the atomistic growth model and the continuum equation [1–4]. Although many growth processes are controlled by diffusion, the Eden model [5], the ballistic deposition model [6], and the restricted solid-on-solid model [7] have a nonlinear growth process that is related to a variety of other different systems such as directed polymer in a random potential [8], Burgers's equation [9], randomly stirred fluids [10] and the Kardar-Parisi-Zhang (KPZ) equation [11].

One of the important quantities in the kinetic roughening of growing surfaces is the surface width W, the root-mean-square fluctuation of the surface height

$$W(L,t)\equiv\left\langle \frac{1}{L^{d_s}}\sum_{\mathbf{x}} [h(\mathbf{x},t)-\overline{h}(t)]^2 \right\rangle^{1/2}, \tag{1}$$

where $h(\mathbf{x},t)$ is the height of site $\mathbf{x}$ at time t. L, $\overline{h}(t)$, and d_s denote the lateral size of the substrate, the mean height at time t, and the substrate dimension (the total dimension $d=d_s+1$), respectively. Here $\langle\ \rangle$ stands for the sample average. The surface width characterizes the roughness of the interface showing a scaling behavior $W\sim L^{\alpha}f(t/L^z)$, where the scaling function $f(x)$ approaches a constant for $x\gg1$, and $f(x)\sim x^{\beta}$ for $x\ll1$ with $z=\alpha/\beta$ [12]. The exponents α, β, and z are called the roughness, the growth, and the dynamic exponent, respectively.

One of the well studied nonlinear equations describing a dynamic growth process is the Kardar-Parisi-Zhang equation [11]

$$\frac{\partial h(\mathbf{x},t)}{\partial t}=\nu\nabla^2 h(\mathbf{x},t)+\frac{\lambda}{2}[\nabla h(\mathbf{x},t)]^2+\eta(\mathbf{x},t) \tag{2}$$

where η is an uncorrelated random noise with strength D. The ballistic deposition (BD) model [6], the restricted solid-on-solid (RSOS) model [7], and the Eden model [5] are generally described by the KPZ equation. For nonzero λ, in $d=2+1$, the equation is controlled by the strong coupling fixed point where $\beta=0.24$–0.25 and $\alpha=0.39$–0.4 are known numerically [7,13]. While for $\lambda=0$ it becomes the Edward-Wilkinson (EW) equation [14], where $\alpha=(3-d)/2$, $\beta=(3-d)/4$, and $z=2$. In $d=2+1$, W^2 grows logarithmically as a function of time. Later, Wolf considered an anisotropic KPZ (AKPZ) equation to study the surface growth on a vicinal substrate [15], where the parallel direction and the perpendicular direction to the vicinal direction are treated differently due to the broken rotational symmetry. This kind of anisotropic equation can be applied to various other problems such as ion-sputtered surface growth [16]. One can write the AKPZ equation

$$\frac{\partial h(\mathbf{x},t)}{\partial t}=\nu_{\perp}\nabla_{\perp}^2 h+\nu_{\parallel}\nabla_{\parallel}^2 h+\frac{\lambda_{\perp}}{2}(\nabla_{\perp}h)^2+\frac{\lambda_{\parallel}}{2}(\nabla_{\parallel}h)^2+\eta, \tag{3}$$

where $\nabla_{\perp}$ ($\nabla_{\parallel}$) denotes the gradient along the perpendicular (parallel) direction [15]. If $\nu_{\perp}=\nu_{\parallel}$ and $\lambda_{\perp}=\lambda_{\parallel}$, Eq. (3) reduces to the isotropic KPZ equation. Using the dynamic renormalization group (RG) method, Wolf has found that when the coefficients of the nonlinear terms have equal signs, the physical properties of the interface are the same as those of the isotropic KPZ equation so there is algebraic roughness. However, when they have opposite signs, the nonlinear terms become irrelevant, and the fixed point is described by the linear EW equation. Also, this remains true when one of the coefficients of the nonlinear terms vanishes. The effects of the anisotropy in three dimensions are studied by both the direct integration of AKPZ equation [17] and the simulations of stochastic lattice models [18,19]. Barabási *et al.* [18] have studied a three-dimensional Toom model that can be described by the AKPZ equation with different amplitudes of $\lambda_{\perp}$ and $\lambda_{\parallel}$ but having the same signs. Jeong *et al.* [19] have introduced a modified Toom model, which is also described by the AKPZ equation with different signs of the nonlinearities.

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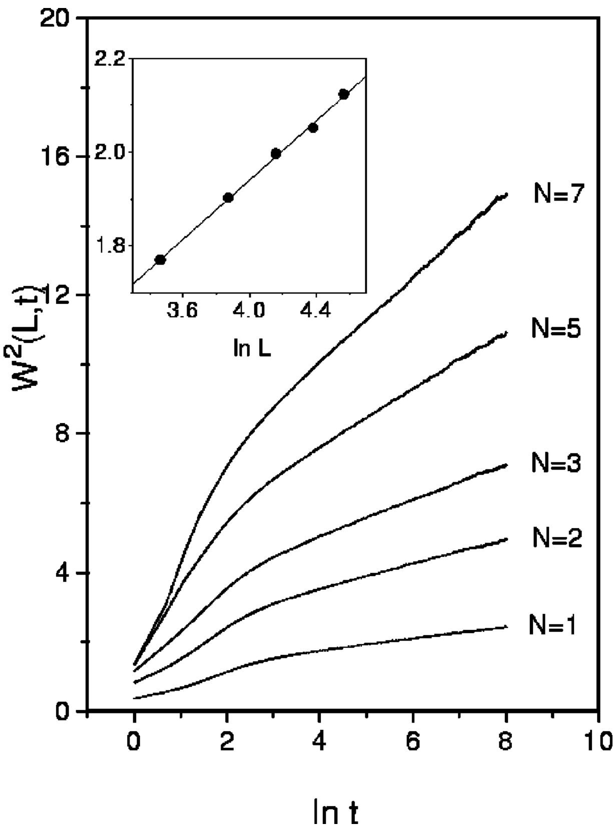

FIG. 1. The semilogarithmic plots of $W^2(L,t)$ vs $\ln t$ for $N=1$, 2,3,5,7, with the system size $L^2=512\times512$, $t=3000$ and 30, independent sample average for the model. The inset shows the semilog plot of $W^2(L)$ vs L with $L=32,48,64,80,96$ at the saturated regime.

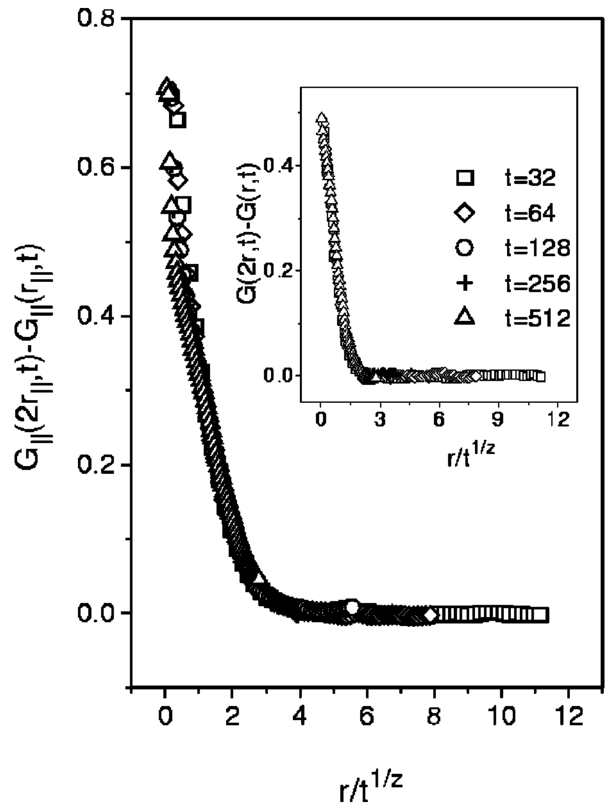

FIG. 2. $G_\parallel(2r_\parallel,t)-G_\parallel(r_\parallel,t)$ as a function of $r_\parallel/\xi_\parallel(t)$ with $\xi_\parallel(t)\sim t^{1/z}$ and $z=2$ for $t=32,64,128,256,512$ and $L^2=512\times512$. The inset shows the plots of $G_\perp(2r_\perp,t)-G_\perp(r_\perp,t)$ as a function of $r_\perp/\xi_\perp(t)$ with $\xi_\perp(t)\sim t^{1/z}$ and $z=2$.

In this paper, we make a composite discrete model to investigate the effect of the anisotropy in the KPZ equation. Our stochastic discrete model is quite different from those in the previous works [18,19]. We are concerned with the anisotropy of the surface morphology, i.e., the ansiotropy generated from two different deposition rules. In $d=2+1$, by applying the BD growth rule in one direction and modified RSOS growth rule in the other direction, $\lambda_\parallel$ and $\lambda_\perp$ of the model can have opposite signs. The width increases logarithmically as a function of time t showing EW type behaviors. We extend our model for the case of higher dimensions and measure the negative values of the exponents. Such an extension in the model is easy and natural compared with the Toom models used before. Another advantage of our model is that both the signs and the values of the nonlinearities $(\lambda_\parallel,\lambda_\perp)$ can be controlled by modifying the growth rules. We also measure the correlation functions for both directions and find $z=2$ to be independent of the directions.

Consider a deposition from the top to the bottom of the two dimensional substrate starting from initially flat surface. Select a site (x,y) randomly. If the surface configuration satisfies the condition

$$h(x,y,t)<h(x\pm1,y,t)+N \tag{4}$$

at time t in the x direction (parallel direction) where N is a preassigned positive integer, then we apply the ballistic deposition rule in the y direction (perpendicular direction) such that $h(x,y,t+1)=\max[h(x,y-1,t),h(x,y,t)+1,h(x,y+1,t)]$ where max takes the maximum value. If the surface configuration does not satisfy the condition [Eq. (4)] in the x direction, in other words, if the height of the selected site exceeds the heights of the nearest neighbor sites by N, the dropped particle is rejected. We call the growth rule, Eq. (4), a generalized restricted solid-on-solid (GRSOS) rule. If we apply the GRSOS rule starting from a flat surface in the one-dimensional substrate, the surface configuration satisfies the RSOS condition which allows only $|h(x+1,t)-h(x,t)|=0,1,\ldots,N$ between the nearest neighbor sites. However, due to the combined growth rules, the developed surface configuration in our model may have the case $|h(x,y,t)-h(x+1,y,t)|>N$, which breaks the restriction on the nearest neighbor height differences in the parallel direction. It is a hybridized model combining both GRSOS and BD rules.

Our simulations are carried out starting from a flat initial condition for $N=1,2,3,5,7$ with a periodic boundary condition in $d=2+1$. We have plotted the surface width W as a function of the logarithmic time for various values of N in Fig. 1 and the surface width W as the function of logarithmic system size in the inset. Our numerical results agree with those of the EW universality class that $W^2\sim\ln t$ at early time for various values of N, and $W^2\sim\ln L$ at the saturated regime. We also extend our model in order to confirm the EW behavior in higher dimensions. In $d=3+1$ and $d=4+1$, we apply the modified RSOS rule in one direction and the BD rule in the other directions. Then the model has negative λ in one direction and positive λ in the other directions. From the RG calculation [19], one would expect EW behaviors with $\alpha=(3-d)/2$. In $d>3$, the EW equation shows $W^2(L)\approx c_0 - c_1 L^{2\alpha}$ where c_0 and c_1 are positive constants. Since α is negative, we need many sample averages to see the correction to c_0. As shown in Fig. 2 we have plotted the square of the surface width $W^2(L)$ as a function of $1/L$ in 3+1 dimensions and $1/L^2$ in 4+1 dimensions [20]. The straight guide lines represent that $\alpha=-\frac{1}{2}$ in 3+1 dimensions and $\alpha=-1$ in 4+1 dimensions, which are the expected values of α in the EW equation. In $d=4+1$, we also apply the modified RSOS rule in two directions and the BD rule in the other two directions and obtain the same results with $\alpha=-1$. So the growth model in higher dimensions also follows the AKPZ equation and our data support the RG results of Jeong *et al.* [19] that even in $d=4$ and 5, the AKPZ equation with opposite signs of the nonlinear terms belongs to the EW class for no space correlated noise.

Hence using the hybridized model combining both GRSOS and BD rules we are able to generate the anisotropy in the growth process, and as a result, the EW behavior has been found. Therefore it is expected that the signs of the nonlinear terms in the model are opposite. We then monitor the average growth velocity $v_\parallel$ and $v_\perp$ as a function of slope m, which is the slope of the tilted surface to find the coefficients of the nonlinear terms [21]. By measuring the average growth velocity v, we can obtain the value of λ through the relation $v(m)\approx v(0)+(\lambda/2)m^2$. Thus, λ can be obtained from the relation $\lambda=(\partial^2 v/\partial m^2)$. We use a helical boundary condition $h(\mathbf{x}\pm L,t)=h(\mathbf{x},t)\pm mL$ with initial vicinal surface of slope m. We have measured v in the parallel and the perpendicular directions and obtained $\lambda_\parallel=-0.61,-0.22,-0.11,-0.04,-0.02$ ($\lambda_\perp=0.22,0.21,0.23,0.25,0.31$) for $N=1$, 2, 3, 5, and 7, respectively. These results support that the model follows the AKPZ equation with different signs of $\lambda_\parallel$ and $\lambda_\perp$. Also the logarithmic behavior of the surface width for the various values of N confirms that the model belongs to the EW class for various values of $\lambda_\parallel/\lambda_\perp=-2.9,-1.05,-0.52,-0.16,-0.06$. Hence if the values of the coefficients of nonlinear terms have opposite signs, the effective coupling constant ($g_\perp=D\lambda_\perp^2/\nu_\perp^3$) flows to zero as shown by the RG calculation [15].

In addition to the surface width, we have calculated the height-height correlation function defined by

$$G(r,t)=\langle[h(\mathbf{x}+\mathbf{r},t)-h(\mathbf{x},t)]^2\rangle_{\mathbf{x}}. \tag{5}$$

The correlation function follows a scaling form $G(r,t)\sim r^{2\alpha}f(r/\xi(t))$ where $r=|\mathbf{r}|$, the correlation length $\xi(t)\sim t^{1/z}$, and the scaling function $f(y)$ approaches constant for $y\ll 1$ and $f(y)\sim y^{-2\alpha}$ for $y\gg 1$. To calculate the correlation functions of the parallel and the perpendicular directions, we define $G_\delta(r,t)=\langle[h(\mathbf{x}+\mathbf{r}_\delta,t)-h(\mathbf{x},t)]^2\rangle_{\mathbf{x}}$ where δ is either $\parallel$ or $\perp$ and r_δ is the distance in δ direction. In 2+1 dimensions, $G(r,t)$ of the EW class has the form $A\ln[rf(r/\xi(t))]$ where A is a constant [22]. To get the data collapse, we monitor $G(2r,t)-G(r,t)$, which should be $A\ln 2+Ag[r/\xi(t)]$ with $g[r/\xi(t)]=\ln f(2r/\xi(t))-\ln f(r/\xi(t))$. As shown in Fig. 3 we have plotted $G_\parallel(2r_\parallel,t)-G_\parallel(r_\parallel,t)$ vs $r_\parallel/t^{1/z}$ and $G_\perp(2r_\perp,t)-G_\perp(r_\perp,t)$ vs $r_\perp/t^{1/z}$ for $t=32,64,128,256,512$. The data are collapsed precisely onto a curve with $z=2$ for both correlation functions, supporting that the two correlation lengths characterizing the morphology of the anisotropic surface follow the same power law. If an anisotropy exponent χ is defined as $\xi_\parallel\sim\xi_\perp^\chi$, then our data support that $\chi=1$ and $\xi_\parallel\sim\xi_\perp\sim t^{1/2}$.

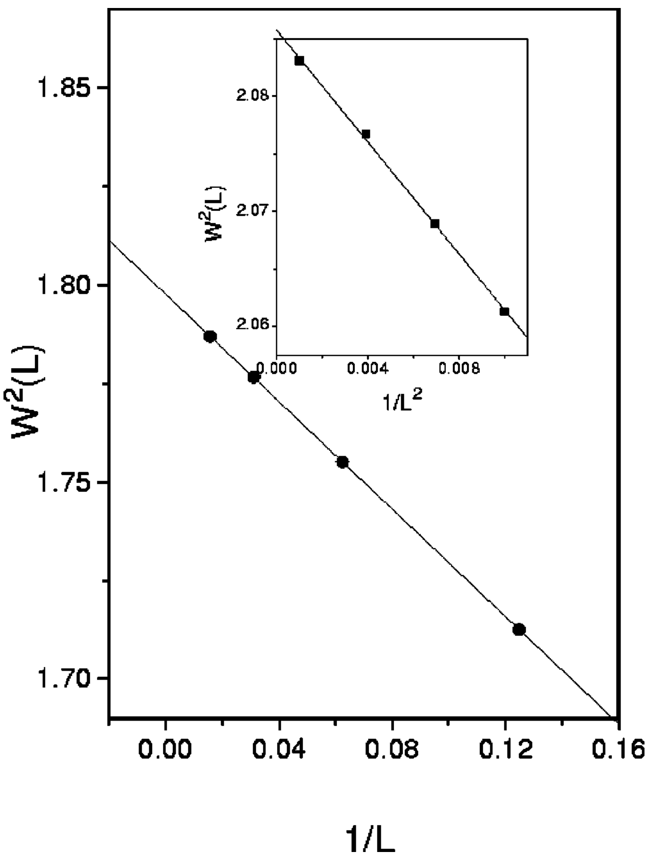

FIG. 3. The plots of $W^2(L)$ vs $1/L$ with $N=1$ and $L^3=8^3,16^3,32^3,64^3$ at the saturated regime in $d=3+1$. The inset shows the plots of $W^2(L,t)$ vs $1/L^2$ with $N=1$ and $L^4=8^4,16^4,32^4$ at the saturated regime in $d=4+1$.

In summary, we introduce a discrete growth model by hybridizing both the ballistic deposition rules in one ($\perp$) direction and the modified RSOS rules in the other ($\parallel$) direction. The former rule allows vacancies in a tilted surface effectively generating positive $\lambda_\perp$ but the latter rule has more rejection rates in a tilted surface producing negative $\lambda_\parallel$. So the model has opposite signs of the nonlinearities and the square of surface width grows logarithmically with time for various values of the nonlinearities in $d=2+1$. We have also introduced the generalized models in $d=3+1$ and $d=4+1$, and have measured the negative values of α successfully. These results completely confirm the EW behavior of our model. As an advantage of the model, the strength of ($\lambda_\parallel/\lambda_\perp$) is controlled by adjusting the restriction parameter N and all results are consistent with the EW class. The measurement of the height-height correlation functions is also carried out to show that both $\xi_\parallel$ and $\xi_\perp$ scales as $t^{1/z}$ with $z=2$.

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